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AN EXTENSION THEOREM FOR PERIODIC TRANSFORMATIONS OF SURFACES. I

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ABSTRACT

For a given simplicial periodic transformation T of the compact orientable surface S, subject to certain constraints on T, a simplicial embedding of S in the 3-dimensional sphere S^3 is defined and an orthogonal periodic transformation t of S^3 such that $t/f(S)$ is equivalent to T is also defined.

1. Introduction

Let $\mathbf{C}^2$ be the 2-dimensional complex space. The 3-dimensional sphere S^3 will be taken as the subset of $\mathbf{C}^2$ defined as

$$S^3 = \{(z_1, z_2) \in \mathbf{C}^2 \mid z_1\bar{z}_1 + z_2\bar{z}_2 = 1\}.$$

Let the transformation $t: S^3 \to S^3$ be defined by

$$t(z_1, z_2) = (w_1 z_1, w_2 z_2); \quad w_k = \exp\left(2\frac{\pi}{n} i d_k\right), \qquad k = 1, 2$$

d_k an integer coprime with n. The number d_k can be 0. The triangulation of S^3 will be as defined in [4].

Furthermore, if S is a compact orientable surface and $T: S \to S$ a simplicial periodic transformation of period n, then T defines in an obvious manner a group action $G(T)$ on S. (We may sometimes write $G(T)$ simply as G.)

1.1. Definition.

$$F(T) = \{x \in S \mid Tx = x\}.$$

This is the fixed point set of $G(T)$.

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1.2. Definition.

$$M(T) = \{x \in S \mid \exists m,\ 1 \leqq m < n \text{ with } T^m x = x\}.$$

The set $M(T)$ we shall choose to call the restricted point set of $G(T)$.

From Definitions 1.1 and 1.2 it follows that $F(T) \subseteq M(T)$. $G(T)$ acts freely on S if $M(T) = \emptyset$.

1.3. Definition (isotropy or stability subgroup). For $x \in S$, the set

$$G_x = \{g \in G \mid gx = x\}$$

is known as the *isotropy* or stability group of x.

1.4. Definition. A subgroup $H \subset G$ will be called an isotropy subgroup if there is an $x \in S$ such that $G_x = H$ or is conjugate to H.

The problem we wish to examine can be stated as follows: given $T: S \to S$ as above, could we find an embedding of S in S^3 together with the transformation $t: S^3 \to S^3$ such that t is an extension of T?

To this we have an answer in the affirmative if T is orientation preserving such that there are at most two isotropy subgroups of $G(T)$ and the orbit of each isotropy subgroup contains even number of elements.

In this paper we shall consider the cases when there is at most one isotropy subgroup. The other cases will be treated in a later paper. Thus, in this paper we shall prove Theorem 3.1 below.

2. The classification theorem for periodic transformations of surfaces

2.1. Definition. Let X and Y be homeomorphic topological spaces and $f_i : X \to Y$, $i = 1, 2$, continuous mappings, then f_1 and f_2 are said to be equivalent if there exists a homeomorphism $g : X \to Y$ such that $f_2 = g f_1 g^{-1}$.

2.2. Lemma. *Let S be an orientable surface, and $T: S \to S$ an orientation preserving periodic transformation with $M(T)$ finite, then for each $x \in M(T)$, there is an integer m, $1 \leqq m < n$ and a neighbourhood D of x such that $T^m \mid D$ is equivalent to the rotation*

$$z \mapsto \exp\left(2 \frac{\pi}{\lambda} id\right) z, \quad \lambda m = n, \lambda \text{ and } d \text{ coprime}.$$

This lemma is proved by Nielsen in [3].

If $\Pi: S \to S/G$, $y \mapsto [y]$ is the projection mapping, then m is the number of elements in $\Pi^{-1}(y)$.

([y] denotes the orbit of y.)

The pair (m, d) may be referred to as the *Valency* of $[y]$ or y.

The following then is the classification theorem as established by Nielsen [3].

2.3. THEOREM. *Two periodic transformations T_1 and T_2 of the surface S are equivalent if and only if they are of the same period n, the sets $M(T_1)/G(T_1)$ and $M(T_2)/G(T_2)$ have equal number of elements and their associated valencies are the same.*

Thus if p is the genus of S and g the genus of $S/G(T)$, we have associated with each transformation T the set of numbers $\{n, p, g, m_1, m_2, \cdots, m_k, d_1, d_2, \cdots, d_k\}$. If $G(T)$ is free then $m_j = 0$, $j = 1, 2, \cdots$.

The parameters $n, p, g, m_1, \cdots, m_k, d_1, d_2, \cdots, d_k$ satisfy the following equations:

(1) $$2p + \sum_{j=1}^{k} m_j - 2 = n(2g + k - 2).$$

This is the Hurwitz equation.

(2) $$d_1 + d_2 + \cdots + d_k \equiv 0 \bmod n.$$

3. The Extension Theorem

3.1. THEOREM. *Let S be a compact orientable surface of genus p and $T: S \to S$ be an orientation preserving* PL *periodic transformation with characteristic parameters*

$$n, p, g, m_1, m_2, \cdots, m_{2\beta}, d_1, d_2, \cdots, d_{2\beta}.$$

If $G(T)$ is free or $m_j = m$ and $d_j = \pm d$, $j = 1, 2, \cdots, 2\beta$, then T has an orthogonal extension to the 3-dimensional sphere.

PROOF. All that is needed is the construction of the embedding of S in S^3 and the extension $t: S^3 \to S^3$ of T.

We divide the construction into three different cases:

(i) $G(T)$ free,

(ii) $m = 1$, i.e. $F(T) = M(T) \neq \varnothing$,

(iii) $1 < m < n$, i.e. $M(T) \neq \varnothing$ but $F(T) = \varnothing$.

Case (i). *$G(T)$ free.* From the Hurwitz condition we have $p = n(g-1) + 1$.

We define $t: S^3 \to S^3$ by

$$t(z_1, z_2) = (wz_1, wz_2), \quad w = \exp\left(\frac{2\pi i}{n}\right).$$

The action $G(T)$ defined by T on S^3 is free. Let the orbit space $S^3/G(t) = L$. The fundamental group $\Pi_1(L, \hat{x})$ is Z_n, $\hat{x} \in L$, $n > 1$.

A t-invariant imbedding of S in S^3

Let w be a non-null homotopic simple closed curve in L. w lifts onto a simple closed curve c in S^3 such that

$$c \cap t^j c = \emptyset, \quad 1 < j < n-1$$

and

$$c \cap t^j c = \{\text{point}\}, \quad j = 1 \text{ or } n-1.$$

Let γ be the simple closed curve

$$\gamma = \bigcup_{j=1}^{n} t^j c.$$

We denote $t^j c$ by c_j, $0 \leqq j \leqq n-1$, $c_0 = c_n$, and the initial and end points of c_j by x_j and x_{j+1} respectively.

Since S^3 and L are triangulated γ may be so chosen such that it belongs to the 1-dimensional skeleton K_1 of the triangulation K or S^3.

We shall define a t-invariant toral tubular neighbourhood V of γ. By adding handles to, or removing blocks from V, we shall obtain a t-variant handle body W (Griffiths [1] and [2]) such that the boundary of W is an embedding of S.

Le H_0 be a tubular neighbourhood of c_0 and $H_j = t^j H_0$, $0 \leqq j \leqq n-1$ such that

$$H_j \cap H_k = \emptyset \quad \text{if } j \neq k \text{ or } k+1 \text{ or } k-1,$$

$$H_j \cap H_{j+1} = D_{j+1}, \quad \text{a disc.}$$

We define $V = \bigcup_{j=0}^{n-1} H_j$. Let

$$E^2 = \{(x_1, x_2) \in R^2 \mid x_1^2 + x_2^2 \leqq 1\} \quad \text{and} \quad I = \{x \in R \mid 0 \leqq x \leqq 1\} = [0, 1].$$

H_j is homeomorphic to $E^2 \times I$. We take the image of c_j under this homeomorphism to be $\{0\} \times I$. Let $h: H_0 \to E^2 \times I$ be the homeomorphism referred to above.

H_0 is now divided into two parts, H_{00} and H_{01}, with

$$h(H_{00}) = E^2 x[0, 1/2] \quad \text{and} \quad h(H_{01}) = E^2 x[1/2, 1].$$

From H_{00} we remove 3-dimensional subsets N_0 and B_1, $0 \leqq 1 \leqq g-1$ as follows (see Fig. 1).

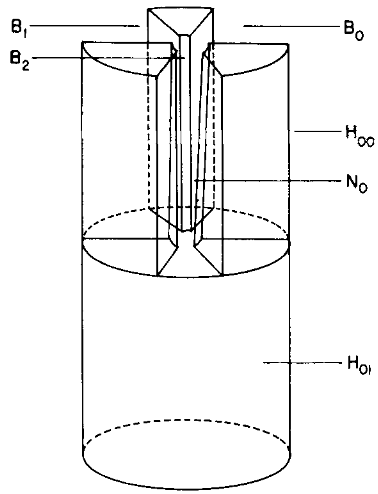

Fig. 1. $g=3$.

If we define $E^2=\{(\rho,\theta)\in R^2 \mid 0\leqq \rho\leqq 1,\ 0\leqq\theta\leqq 2\pi\}$, then

$$N_0=\{(\rho,\theta)\in E^2 \mid 0\leqq\rho\leqq 1/2\}\times I_{1/2},\quad I_{1/2}=[0,1/2]\quad\text{and}$$

$$B_1=\left\{(\rho,\theta)\in E^2 \mid 1/2\leqq\rho\leqq 1,\ \frac{2\pi l}{g}\leqq\theta\leqq 2\pi\frac{(2l+1)}{2g}\right\}\times I_{1/2}.$$

The set $\mathrm{Cl}(V-N_0\bigcup_{l=0}^{g-1}B_1)$ is a handle body of genus g ($\mathrm{Cl}X$ denotes the closure of the set X).

We now define

$$W=\mathrm{Cl}\left(V-\bigcup_{j=0}^{n-1}t^j\left(N_0\bigcup_{l=0}^{g-1}B_l\right)\right);$$

W is the required handle body with genus p where

$$p=n(g-1)+1 \tag{3}$$

(see Fig. 2). Then if ∂W denotes the boundary of W, ∂W is homeomorphic with S and $t \mid \partial W$ is equivalent with T — since the equivalence class of T is uniquely determined by Eq. (3) above.

Case (ii). $m=1$, *i.e.* $F(T)=M(T)\neq\emptyset$. We define $t:S^3\to S^3$ by

$$t(z_1,z_2)=(wz_1,z_2)$$

where $w=\exp(2\pi id/n)$ without loss of generality; we may assume $d=1$. By the Hurwitz condition we have

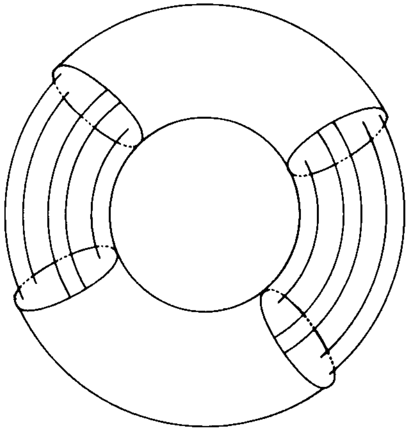

Fig. 2. $n = 2$, $g = 3$ and $p = 5$.

$$p = (n-1)(g+\beta-1)+g. \tag{4}$$

The fixed point set $F(t) = \{(0, z_2) \in S^3\}$. This is homeomorphic with the unit circle S^1.

Let V be a t-invariant toral tubular neighbourhood of $F(t)$. We shall now construct from V a t-invariant handle body of genus p.

$V \approx S^1 \times E^2$. Now let $I_\pi = \{z \in S^1 \mid x \leqq \arg z \leqq \pi\}$ and $V_0 = I_\pi \times E^2$. We divide V_0 into $2(g+\beta-1)+1$ equal parts H_j, $0 \leqq j \leqq 2(g+\beta-1)$, where $H_j = I_j \times E^2$ and I_j is the closed interval

$$I_j = \left[\frac{\pi j}{2(g+\beta-1)+1}, \frac{\pi(j+1)}{2(g+\beta-1)+1}\right].$$

From each H_{2j+1} we remove subsets N_{2j+1} of the type N_0 in case (i) and subsets $B_{1,2j+1}$ of the type B_1 in case (i), where

$$N_{2j+1} = \{(\rho, \theta) \in E^2 \mid 0 \leqq \rho \leqq 1/2\} \times I_{2j+1},$$

$$B_{1,2j+1} = \left\{(\rho, \theta) \in E^2 \,\middle|\, 1/2 \leqq \rho \leqq 1, \frac{2\pi l}{n} \leqq \theta \leqq \frac{2\pi(2l+1)}{2n}\right\} \times I_{2j+1}.$$

Furthermore, choose g of the H_{2k} $(0 \leqq k \leqq (\beta+g-1))$ and remove from each H_{2k} the core N_{2k}, where $N_{2k} = \{(\rho, \theta) \in E^2 \mid 0 \leqq \rho \leqq 1/2\} \times I_{2k}$.

Let $N_1^1 = \bigcup_{j=0}^{g+\beta-2} N_{2j+1}$, $N_0^1 = \bigcup_{k=0}^{g-1} N_{2k}$ and $B = \bigcup_{l=0}^{n-1} (\bigcup_{j=0}^{g+\beta-1} B_{l,2j+1})$, then the subset $W = \mathrm{Cl}(V_0 - [N_1^1 \cup B \cup N_0])$ of S^3 is a handle body with p handles where $p = (n-1)(g+\beta-1)+g$. If $S = \partial W$ then $tS = S$ and S is t-invariant. This is the required embedding of S in S^3 (Fig. 3).

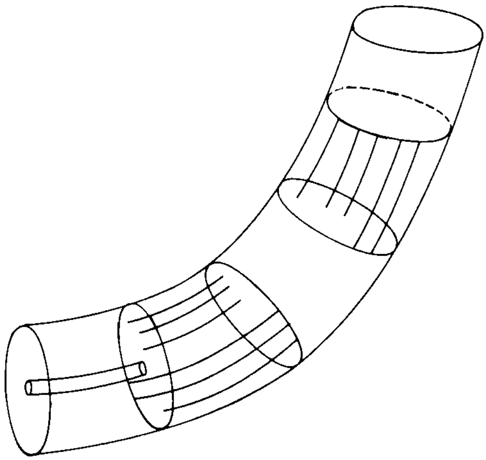

Fig. 3. $p = 7$, $n = 4$, $g = 1$, $\beta = 2$ and $m = 1$.

Case (iii). $1 < m < n$, *i.e.* $M(T) \neq \emptyset$ *but* $F(T) = \emptyset$. From the Hurwitz condition we have

$$p + m\beta - 1 = n(g + \beta - 1). \tag{5}$$

Again we take $d = \pm 1$. We define $t: S^3 \to S^3$ by

$$t(z_1, z_2) = (w_1 z_1, w_2 z_2),$$

$w_1 = \exp(2\pi i/m)$ and $w_2 = \exp(2\pi i/n)$. Then $F(t) = \emptyset$, $M(t) = F(t^m) = \{(z_1, 0) \in S^3\} \approx S^1$ and the tubular neighbourhood V exists with the required properties. Hence let $V \subset S^3$ be a t-invariant toral tubular neighbourhood of $F(t^m)$.

$V \approx S^1 \times E^2$. We now modify V as follows: from condition (5) we have

$$p - 1 = n(g - 1) + m\beta(\lambda - 1), \quad \text{where } \lambda m = n. \tag{6}$$

We cut V into $2m\beta$ parts such that

$$V = \bigcup_{j=0}^{2m\beta} H_j, \qquad t^k H_j \in \{H_0, H_1, \cdots, H_{2m\beta}\},$$

$$k = 0, 1, 2, \cdots, n - 1, \qquad j = 0, 1, \cdots, 2m\beta,$$

$$H_j = I_j \times E^2, \qquad I_j = \left[\frac{2\pi j}{2m\beta}, \frac{2\pi(j+1)}{2m\beta}\right].$$

We delete from V the subset

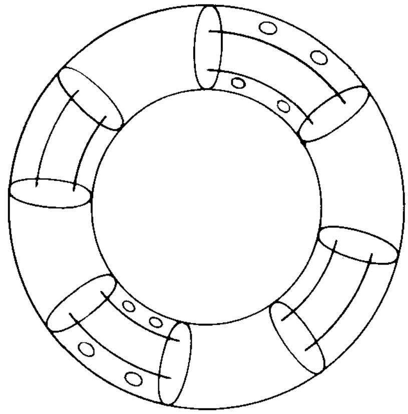

Fig. 4. $p = 13$, $n = 4$, $m = 2$, $\beta = 2$ and $g = 3$.

$$\bigcup_{j=0}^{m\beta-1} H_{2j}, \qquad t^k H_{2j} \in \{H_0, H_2, \cdots, H_{2m\beta}\},$$

and as in cases (i) and (ii) we attach $\lambda - 1$ handles to connect H_{2j+1} with H_{2j+3}, $1 \leqq j \leqq m\beta - 1$. So far we have attached $m\beta(\lambda - 1)$ handles to $\mathrm{Cl}(V - \cup H_{2j})$. (In effect we connect H_{2j+1} with H_{2j+3} with λ solid cylinders, D_{jl}, $0 \leqq l \leqq \lambda - 1$.)

Let the λ solid cylinders connecting $H_{2m\beta-1}$ and H_1 be D_{0l}, $0 \leqq l \leqq \lambda - 1$, where

$$D_{0l} = \left\{(\rho, \theta) \in E^2 \,\middle|\, 1/2 \leqq \rho \leqq 1, \frac{2\pi l}{\lambda} \leqq \theta \leqq 2\pi \frac{(2l+1)}{2\lambda}\right\} \times I_0.$$

If on D_{00} *we remove* $g - 1$ blocks B_k, such that $\mathrm{Cl}(D_{00} - \bigcup_{k=0}^{g-2} B_k)$ is a connected handle body with $g - 1$ handles, then we would have completed the construction by allowing t to remove from $t^i D_{00}$ the blocks $t^i(\bigcup_k B_k)$, $0 \leqq i \leqq n - 1$, $0 \leqq k \leqq g - 2$.

We define $B_k \subset D_{00}$ as follows:

$$B_k = \left\{(\rho, \theta) \in E^2 \,\middle|\, 1/2 \leqq \rho \leqq 1, \frac{2\pi}{6\lambda} \leqq \theta \leqq \frac{4\pi}{6\lambda}\right\} \times I_{0k},$$

$$I_{0k} = \left[\frac{2\pi}{2m\beta}\,\frac{2k}{2(g-1)}, \frac{2\pi}{2m\beta}\,\frac{2k+1}{2(g-1)}\right], \qquad 0 \leqq k \leqq g - 2.$$

Let $B = \bigcup_{i=0}^{n-1} t^i(\bigcup_{k=0}^{g-2} B_k)$, $H = \bigcup_{j=0}^{m\beta-1} H_{2j}$ and $D = \bigcup_{j=0}^{m\beta-1}(\bigcup_{l=0}^{\lambda-1} D_{jl})$. We now define the handle body W as

$$W = \mathrm{Cl}((V - H) \cup (D - B)).$$

By construction W is a handle body with p handles where

$$p = n(g-1) + m\beta(\lambda - 1) + 1 \tag{7}$$

(the extra handle is for closing up $\mathrm{Cl}(V - H)$). Then again by Nielsen's theorem S is homeomorphic to W and $t \mid \partial W$ is equivalent to T (Fig. 4). This completes the proof of our Theorem 3.1.

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